Solved Problems

5.1 Let

$$s_x = \frac{vx_{\max} - vx_{\min}}{wx_{\max} - wx_{\min}} \quad \text{and} \quad s_y = \frac{vy_{\max} - vy_{\min}}{wy_{\max} - wy_{\min}}$$

Express window-to-viewport mapping in the form of a composite transformation matrix.

SOLUTION

$$N = \begin{pmatrix} 1 & 0 & vx_{\min} \\ 0 & 1 & vy_{\min} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & -wx_{\min} \\ 0 & 1 & -wy_{\min} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_x & 0 & -s_x wx_{\min} + vx_{\min} \\ 0 & s_y & -s_y wy_{\min} + vy_{\min} \\ 0 & 0 & 1 \end{pmatrix}$$

5.2 Find the normalization transformation that maps a window whose lower left corner is at $(1, 1)$ and upper right corner is at $(3, 5)$ onto (a) a viewport that is the entire normalized device screen and (b) a viewport that has lower left corner at $(0, 0)$ and upper right corner $(\frac{1}{2}, \frac{1}{2})$.

SOLUTION

From Prob. 5.1, we need only identify the appropriate parameters.

(a) The window parameters are $wx_{\min} = 1$, $wx_{\max} = 3$, $wy_{\min} = 1$, and $wy_{\max} = 5$. The viewport parameters are $vx_{\min} = 0$, $vx_{\max} = 1$, $vy_{\min} = 0$, and $vy_{\max} = 1$. Then $s_x = \frac{1}{2}$, $s_y = \frac{1}{4}$, and

$$N = \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

(b) The window parameters are the same as in (a). The viewport parameters are now $vx_{\min} = 0$, $vx_{\max} = \frac{1}{2}$, $vy_{\min} = 0$, $vy_{\max} = \frac{1}{2}$. Then $s_x = \frac{1}{4}$, $s_y = \frac{1}{8}$, and

$$N = \begin{pmatrix} \frac{1}{4} & 0 & -\frac{1}{4} \\ 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 1 \end{pmatrix}$$

5.3 Find the complete viewing transformation that maps a window in world coordinates with x extent 1 to 10 and y extent 1 to 10 onto a viewport with x extent $\frac{1}{4}$ to $\frac{3}{4}$ and y extent 0 to $\frac{1}{2}$ in normalized device space, and then maps a workstation window with x extent $\frac{1}{4}$ to $\frac{1}{2}$ and y extent $\frac{1}{4}$ to $\frac{1}{2}$ in the normalized device space into a workstation viewport with x extent 1 to 10 and y extent 1 to 10 on the physical display device.

SOLUTION

From Prob. 5.1, the parameters for the normalization transformation are $wx_{\min} = 1$, $wx_{\max} = 10$, $wy_{\min} = 1$, $wy_{\max} = 10$, and $vx_{\min} = \frac{1}{4}$, $vx_{\max} = \frac{3}{4}$, $vy_{\min} = 0$, and $vy_{\max} = \frac{1}{2}$. Then

$$s_x = \frac{1/2}{9} = \frac{1}{18} \qquad s_y = \frac{1/2}{9} = \frac{1}{18}$$

and

$$N = \begin{pmatrix} \frac{1}{18} & 0 & \frac{7}{36} \\ 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 0 & 1 \end{pmatrix}$$

The parameters for the workstation transformation are $wx_{\min} = \frac{1}{4}$, $wx_{\max} = \frac{1}{2}$, $wy_{\min} = \frac{1}{4}$, $wy_{\max} = \frac{1}{2}$, and $vx_{\min} = 1$, $vx_{\max} = 10$, $vy_{\min} = 1$, and $vy_{\max} = 10$. Then

$$s_x = \frac{9}{1/4} = 36 \qquad s_y = \frac{9}{1/4} = 36$$

and

$$W = \begin{pmatrix} 36 & 0 & -8 \\ 0 & 36 & -8 \\ 0 & 0 & 1 \end{pmatrix}$$

The complete viewing transformation V is

$$V = W \cdot N = \begin{pmatrix} 36 & 0 & -8 \\ 0 & 36 & -8 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{18} & 0 & \frac{7}{36} \\ 0 & \frac{1}{18} & -\frac{1}{18} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -10 \\ 0 & 0 & 1 \end{pmatrix}$$

5.4 Find a normalization transformation from the window whose lower left corner is at $(0, 0)$ and upper right corner is at $(4, 3)$ onto the normalized device screen so that aspect ratios are preserved.

SOLUTION

The window aspect ratio is $a_w = \frac{4}{3}$. Unless otherwise indicated, we shall choose a viewport that is as large as possible with respect to the normalized device screen. To this end, we choose the x extent from 0 to 1 and the y extent from 0 to $\frac{3}{4}$. So

$$a_v = \frac{1}{3/4} = \frac{4}{3}$$

As in Prob. 5.2, with parameters $wx_{\min} = 0$, $wx_{\max} = 4$, $wy_{\min} = 0$, $wy_{\max} = 3$ and $vx_{\min} = 0$, $vx_{\max} = 1$, $vy_{\min} = 0$, $vy_{\max} = \frac{3}{4}$,

$$N = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

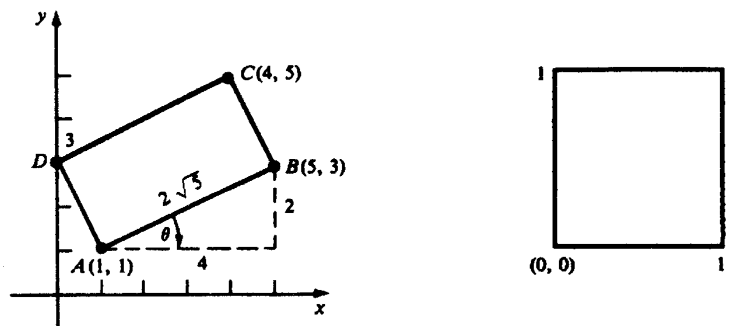
5.5 Find the normalization transformation N which uses the rectangle $A(1, 1)$, $B(5, 3)$, $C(4, 5)$, $D(0, 3)$ as a window [Fig. 5-16(a)] and the normalized device screen as a viewport [Fig. 5-16(b)].

SOLUTION

We will first rotate the rectangle about A so that it is aligned with the coordinate axes. Next, as in Prob. 5.1, we calculate s_x and s_y and finally we compose the rotation and the transformation N (from Prob. 5.1) to find the required normalization transformation N_R.

The slope of the line segment $\overline{AB}$ is

$$m = \frac{3-1}{5-1} = \frac{1}{2}$$



(a) Window. (b) Viewport.

Fig. 5-16

Looking at Fig. 5-11, we see that $-\theta$ will be the direction of the rotation. The angle θ is determined from the slope of a line (App. 1) by the equation $\tan\theta = \frac{1}{2}$. Then

$$\sin\theta = \frac{1}{\sqrt{5}}, \quad \text{and so} \quad \sin(-\theta) = -\frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}, \quad \cos(-\theta) = \frac{2}{\sqrt{5}}$$

The rotation matrix about $A(1, 1)$ is then (Chap. 4, Prob. 4.4):

$$R_{-\theta,A} = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & \left(1 - \frac{3}{\sqrt{5}}\right) \\ -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & \left(1 - \frac{1}{\sqrt{5}}\right) \\ 0 & 0 & 1 \end{pmatrix}$$

The x extent of the rotated window is the length of $\overline{AB}$. Similarly, the y extent is the length of $\overline{AD}$. Using the distance formula (App. 1) to calculate these lengths yields

$$d(A, B) = \sqrt{2^2 + 4^2} = \sqrt{20} = 2\sqrt{5} \qquad d(A, D) = \sqrt{1^2 + 2^2} = \sqrt{5}$$

Also, the x extent of the normalized device screen is 1, as is the y extent. Calculating s_x and s_y,

$$s_x = \frac{\text{viewport } x \text{ extent}}{\text{window } x \text{ extent}} = \frac{1}{2\sqrt{5}} \qquad s_y = \frac{\text{viewport } y \text{ extent}}{\text{window } y \text{ extent}} = \frac{1}{\sqrt{5}}$$

So

$$N = \begin{pmatrix} \frac{1}{2\sqrt{5}} & 0 & -\frac{1}{2\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 0 & 0 & 1 \end{pmatrix}$$

The normalization transformation is then

$$N_R = N \cdot R_{-\theta,A} = \begin{pmatrix} \frac{1}{5} & \frac{1}{10} & -\frac{3}{10} \\ -\frac{1}{5} & \frac{2}{5} & -\frac{1}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

5.6 Let R be the rectangular window whose lower left-hand corner is at $L(-3, 1)$ and upper right-hand corner is at $R(2, 6)$. Find the region codes for the endpoints in Fig. 5-17.

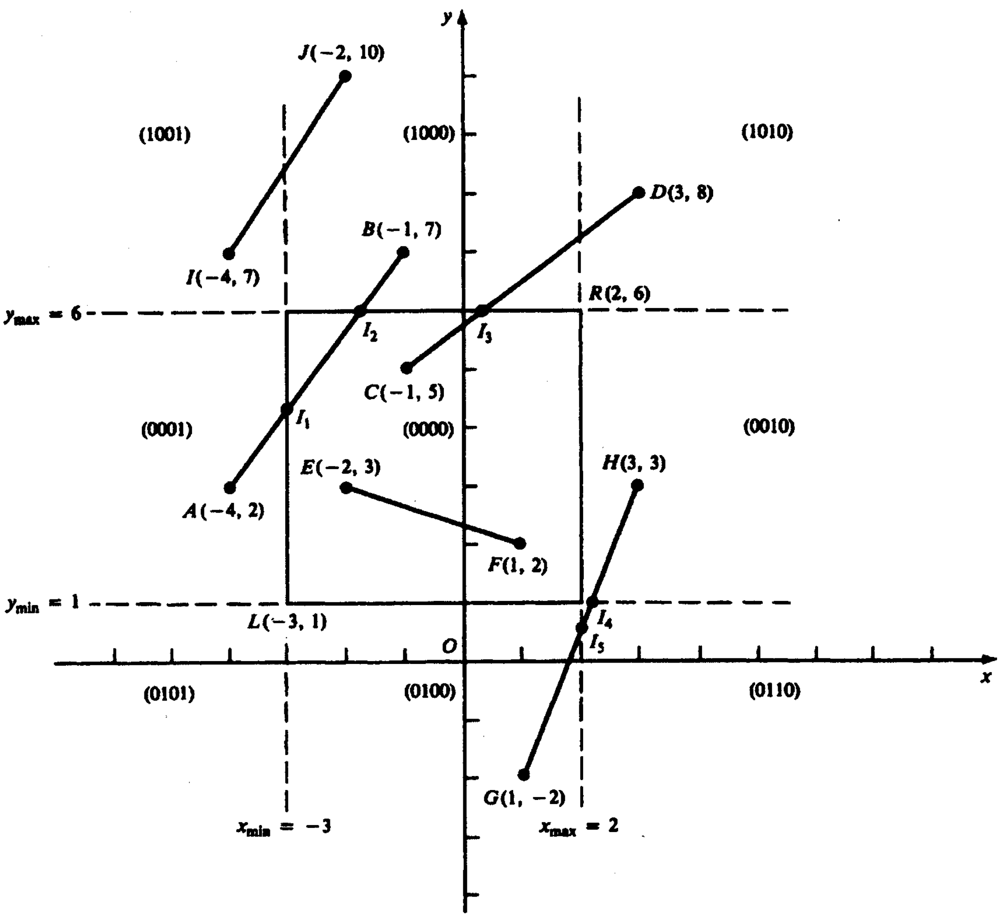


Fig. 5-17

SOLUTION

The region code for point (x, y) is set according to the scheme

$$\text{Bit } 1 = \text{sign}(y - y_{max}) = \text{sign}(y - 6) \qquad \text{Bit } 3 = \text{sign}(x - x_{max}) = \text{sign}(x - 2)$$
$$\text{Bit } 2 = \text{sign}(y_{min} - y) = \text{sign}(1 - y) \qquad \text{Bit } 4 = \text{sign}(x_{min} - x) = \text{sign}(-3 - x)$$

Here

$$\text{Sign}(a) = \begin{cases} 1 & \text{if } a \text{ is positive} \\ 0 & \text{otherwise} \end{cases}$$

So

$$\begin{array}{ll} A(-4, 2) \rightarrow 0001 & F(1, 2) \rightarrow 0000 \\ B(-1, 7) \rightarrow 1000 & G(1, -2) \rightarrow 0100 \\ C(-1, 5) \rightarrow 0000 & H(3, 3) \rightarrow 0010 \\ D(3, 8) \rightarrow 1010 & I(-4, 7) \rightarrow 1001 \\ E(-2, 3) \rightarrow 0000 & J(-2, 10) \rightarrow 1000 \end{array}$$

5.8 Find the clipping categories for the line segments in Prob. 5.6 (see Fig. 5-17).

SOLUTION

We place the line segments in their appropriate categories by testing the region codes found in Prob. 5.6.

Category 1 (visible): $\overline{EF}$ since the region code for both endpoints is 0000.

Category 2 (not visible): $\overline{IJ}$ since (1001) AND (1000) = 1000 (which is not 0000).

Category 3 (candidates for clipping): $\overline{AB}$ since (0001) AND (1000) = 0000, $\overline{CD}$ since (0000) AND (1010) = 0000, and $\overline{GH}$ since (0100) AND (0010) = 0000.

5.9 Use the Cohen–Sutherland algorithm to clip the line segments in Prob. 5.6 (see Fig. 5-17).

SOLUTION

From Prob. 5.8, the candidates for clipping are $\overline{AB}$, $\overline{CD}$, and $\overline{GH}$.

In *clipping* $\overline{AB}$, the code for A is 0001. To push the 1 to 0, we clip against the boundary line $x_{min} = -3$. The resulting intersection point is $I_1(-3, 3\frac{2}{3})$. We clip (do not display) $\overline{AI_1}$ and work on $\overline{I_1B}$. The code for I_1 is 0000. The clipping category for $\overline{I_1B}$ is 3 since (0000) AND (1000) is (0000). Now B is outside the window (i.e., its code is 1000), so we push the 1 to a 0 by clipping against the line $y_{max} = 6$. The resulting intersection is $I_2(-1\frac{3}{5}, 6)$. Thus $\overline{I_2B}$ is clipped. The code for I_2 is 0000. The remaining segment $\overline{I_1I_2}$ is displayed since both endpoints lie in the window (i.e., their codes are 0000).

For *clipping* $\overline{CD}$, we start with D since it is outside the window. Its code is 1010. We push the first 1 to a 0 by clipping against the line $y_{max} = 6$. The resulting intersection I_3 is $(\frac{1}{3}, 6)$ and its code is 0000. Thus $\overline{I_3D}$ is clipped and the remaining segment $\overline{CI_3}$ has both endpoints coded 0000 and so it is displayed.

For *clipping* $\overline{GH}$, we can start with either G or H since both are outside the window. The code for G is 0100, and we push the 1 to a 0 by clipping against the line $y_{min} = 1$. The resulting intersection point is $I_4(2\frac{1}{5}, 1)$, and its code is 0010. We clip $\overline{GI_4}$ and work on $\overline{I_4H}$. Segment $\overline{I_4H}$ is not displayed since (0010) AND (0010) = 0010.

5.10 Clip line segment $\overline{CD}$ of Prob. 5.6 by using the midpoint subdivision process.

SOLUTION

The midpoint subdivision process is based on repeated bisections. To avoid continuing indefinitely, we agree to say that a point (x_1, y_1) lies on any of the boundary lines of the rectangle, say, boundary line $x = x_{max}$, for example, if $-TOL \leq x_1 - x_{max} \leq TOL$. Here TOL is a prescribed tolerance, some small number, that is set before the process begins.

To clip $\overline{CD}$, we determine that it is in category 3. For this problem we arbitrarily choose TOL = 0.1. We find the midpoint of $\overline{CD}$ to be $M_1(1, 6.5)$. Its code is 1000.

So $\overline{M_1D}$ is not displayed since (1000) AND (1010) = 1000. We further subdivide $\overline{CM_1}$ since (0000) AND (1000) = 0000. The midpoint of $\overline{CM_1}$ is $M_2(0, 5.75)$; the code for M_2 is 0000. Thus $\overline{CM_2}$ is displayed since both endpoints are 0000 and $\overline{M_2M_1}$ is a candidate for clipping. The midpoint of $\overline{M_2M_1}$ is $M_3(0.5, 6.125)$, and its code is 1000. Thus $\overline{M_3M_1}$ is chipped and $\overline{M_2M_3}$ is subdivided. The midpoint of $\overline{M_2M_3}$ is $M_4(0.25, 5.9375)$, whose code is 0000. However, since $y_1 = 5.9375$ lies within the tolerance 0.1 of the boundary line $y_{max} = 6$—that is, $6 - 5.9375 = 0.0625 < 0.1$, we agree that M_4 lies on the boundary line $y_{max} = 6$. Thus $\overline{M_2M_4}$ is displayed and $\overline{M_4M_3}$ is not displayed. So the original line segment $\overline{CD}$ is clipped at M_4 and the process stops.

5.11 Suppose that in an implementation of the Cohen–Sutherland algorithm we choose boundary lines in the top–bottom–right–left order to clip a line in category 3, draw a picture to show a worst-case scenario, i.e., one that involves the highest number of iterations.

SOLUTION

See Fig. 5-18.

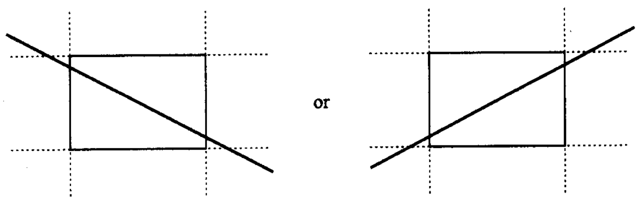


Fig. 5-18

5.12 Use the Liang–Barsky algorithm to clip the lines in Fig. 5-19.

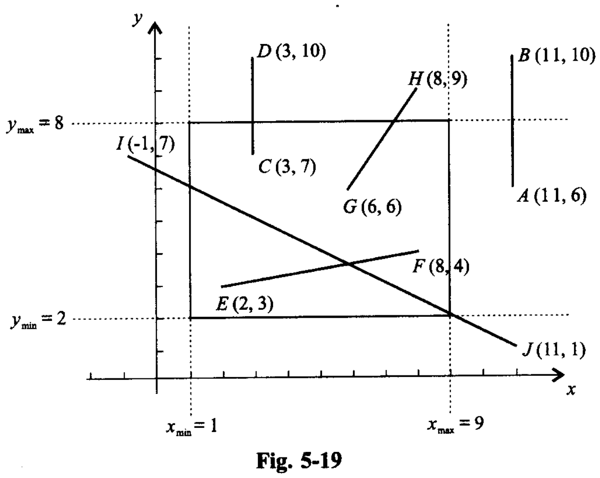


Fig. 5-19

SOLUTION

For line AB, we have

$$\begin{array}{ll} p_1 = 0 & q_1 = 10 \\ p_2 = 0 & q_2 = -2 \\ p_3 = -4 & q_3 = 4 \\ p_4 = 4 & q_4 = 2 \end{array}$$

Since $p_2 = 0$ and $q_2 < -2$, AB is completely outside the right boundary.

For line CD, we have

$$\begin{array}{lll} p_1 = 0 & q_1 = 2 & \\ p_2 = 0 & q_2 = 6 & \\ p_3 = -3 & q_3 = 5 & r_3 = -\frac{5}{3} \\ p_4 = 3 & q_4 = 1 & r_4 = \frac{1}{3} \end{array}$$

Thus $u_1 = \max(0, -\frac{5}{3}) = 0$ and $u_2 = \min(1, \frac{1}{3}) = \frac{1}{3}$. Since $u_1 < u_2$, the two endpoints of the clipped line are $(3, 7)$ and $(3, 7 + 3(\frac{1}{3})) = (3, 8)$.

For line EF, we have

$$\begin{array}{lll} p_1 = -6 & q_1 = 1 & r_1 = -\frac{1}{6} \\ p_2 = 6 & q_2 = 7 & r_2 = \frac{7}{6} \\ p_3 = -1 & q_3 = 1 & r_3 = -\frac{1}{1} \\ p_4 = 1 & q_4 = 5 & r_4 = \frac{5}{1} \end{array}$$

Thus $u_1 = \max(0, -\frac{1}{6}, -1) = 0$ and $u_2 = \min(1, \frac{7}{6}, 5) = 1$. Since $u_1 = 0$ and $u_2 = 1$, line EF is completely inside the clipping window.

For line GH, we have

$$\begin{array}{lll} p_1 = -2 & q_1 = 5 & r_1 = -\frac{5}{2} \\ p_2 = 2 & q_2 = 3 & r_2 = \frac{3}{2} \\ p_3 = -3 & q_3 = 4 & r_3 = -\frac{4}{3} \\ p_4 = 3 & q_4 = 2 & r_4 = \frac{2}{3} \end{array}$$

Thus $u_1 = \max(0, -\frac{5}{2}, -\frac{4}{3}) = 0$ and $u_2 = \min(1, \frac{3}{2}, \frac{2}{3}) = \frac{2}{3}$. Since $u_1 < u_2$, the two endpoints of the clipped line are $(6, 6)$ and $(6 + 2(\frac{2}{3}), 6 + 3(\frac{2}{3})) = (7\frac{1}{3}, 8)$.

For line IJ, we have

$$\begin{array}{lll} p_1 = -12 & q_1 = -2 & r_1 = \frac{1}{6} \\ p_2 = 12 & q_2 = 10 & r_2 = \frac{5}{6} \\ p_3 = 6 & q_3 = 5 & r_3 = \frac{5}{6} \\ p_4 = -6 & q_4 = 1 & r_4 = -\frac{1}{6} \end{array}$$

Thus $u_1 = \max(0, \frac{1}{6}, -\frac{1}{6}) = \frac{1}{6}$ and $u_2 = \min(1, \frac{5}{6}, \frac{5}{6}) = \frac{5}{6}$. Since $u_1 < u_2$, the two endpoints of the clipped line are $(-1 + 12(\frac{1}{6}), 7 + (-6)(\frac{1}{6})) = (1, 6)$ and $(-1 + 12(\frac{5}{6}), 7 + (-6)(\frac{5}{6})) = (9, 2)$.

5.13 How can we determine whether a point $P(x, y)$ lies to the left or to the right of a line segment joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$?

SOLUTION

Refer to Fig. 5-20. Form the vectors **AB** and **AP**. If the point P is to the left of **AB**, then by the definition of the cross product of two vectors (App. 2) the vector **AB** $\times$ **AP** points in the direction of the vector **K** perpendicular to the xy plane (see Fig. 5-20). If it lies to the right, the cross product points in the direction

$-\mathbf{K}$. Now

$$\mathbf{AB} = (x_2 - x_1)\mathbf{I} + (y_2 - y_1)\mathbf{J} \qquad \mathbf{AP} = (x - x_1)\mathbf{I} + (y - y_1)\mathbf{J}$$

So

$$\mathbf{AB} \times \mathbf{AP} = [(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)]\mathbf{K}$$

Then the direction of this cross product is determined by the number

$$\bar{C} = (x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)$$

If $\bar{C}$ is positive, P lies to the left of **AB**. If $\bar{C}$ is negative, then P lies to the right of **AB**.

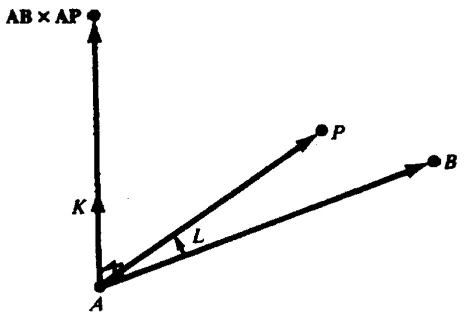


Fig. 5-20

5.14 Draw a flowchart illustrating the logic of the Sutherland–Hodgman algorithm.

SOLUTION

The algorithm inputs the vertices of a polygon one at a time. For each input vertex, either zero, one, or two output vertices will be generated depending on the relationship of the input vertices to the clipping edge E.

We denote by P the input vertex, S the previous input vertex, and F the first arriving input vertex. The vertex or vertices to be output are determined according to the logic illustrated in the flowchart in Fig. 5-21. Recall that a polygon with n vertices $P_1, P_2, \ldots, P_n$ has n edges $\overline{P_1P_2}, \ldots, \overline{P_{n-1}P_n}$ and the edge $\overline{P_nP_1}$ closing the polygon. In order to avoid the need to duplicate the input of P_1 as the final input vertex (and a corresponding mechanism to duplicate the final output vertex to close the polygon), the closing logic shown in the flowchart in Fig. 5-22 is called after processing the final input vertex P_n.

5.15 Clip the polygon $P_1, \ldots, P_9$ in Fig. 5-23 against the window $ABCD$ using the Sutherland–Hodgman algorithm.

SOLUTION

At each stage the new output polygon, whose vertices are determined by applying the Sutherland–Hodgman algorithm (Prob. 5.14), is passed on to the next clipping edge of the window $ABCD$. The results are illustrated in Figs. 5-24 through 5-27.

5.16 Clip the polygon $P_1, \ldots, P_8$ in Fig. 5-10 against the rectangular clipping window using the Sutherland–Hodgman algorithm.

SOLUTION

We first clip against the top boundary line, then the left, and finally the bottom. The right boundary is omitted since it does not affect any vertex list. The intermediate and final results are in Fig. 5-28.

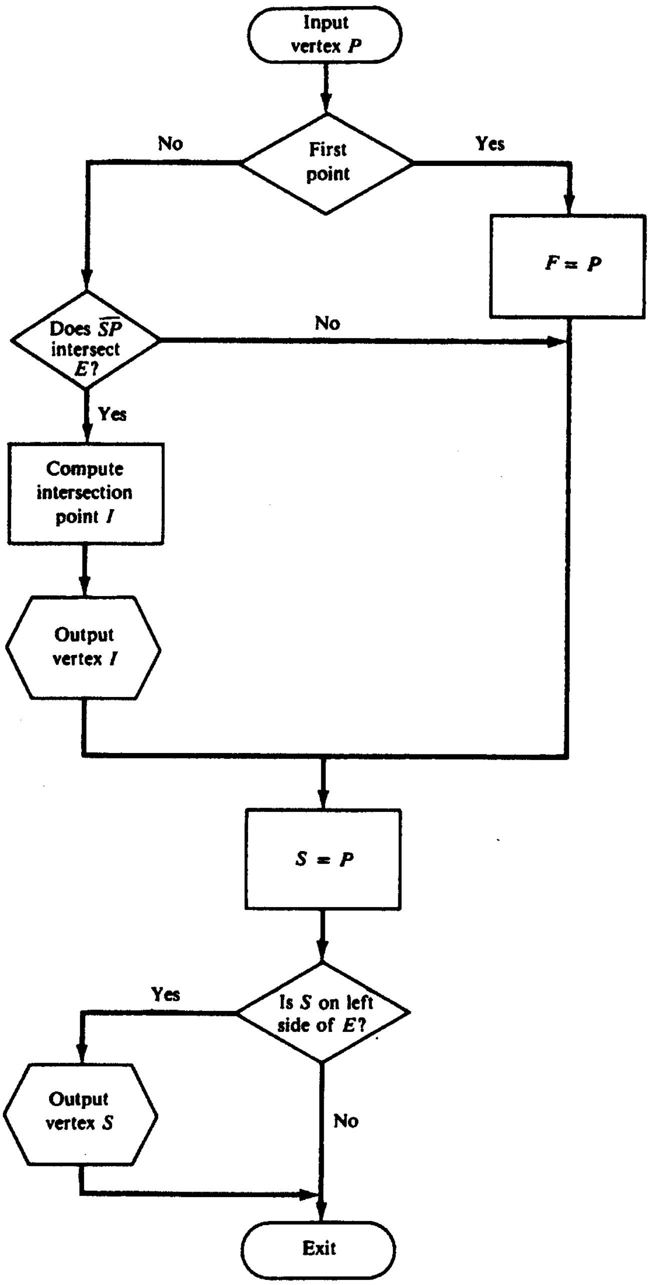

Input
vertex P
First
point
No
Yes
F = P
Does SP intersect E?
No
Yes
Compute
intersection
point I
Output
vertex I
S = P
Yes
Is S on left
side of E?
No
Output
vertex S
Exit

Fig. 5-21

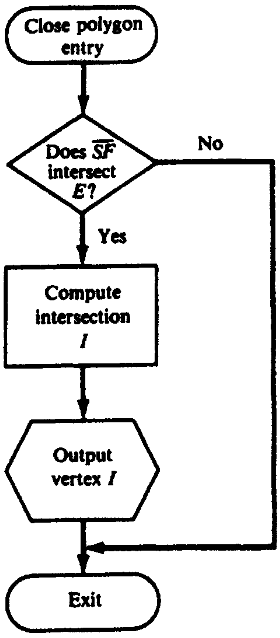


Fig. 5-22

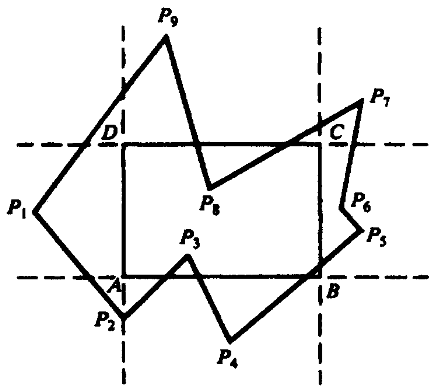


Fig. 5-23

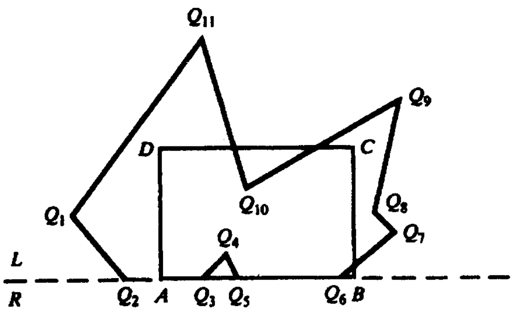


Fig. 5-24 Clip against $\overline{AB}$.

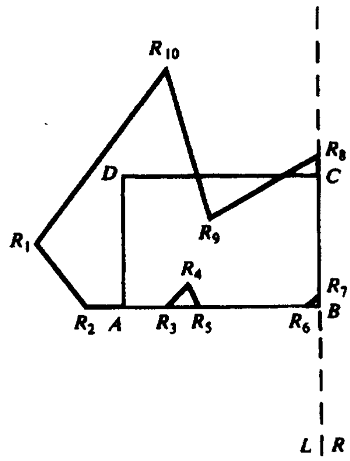


Fig. 5-25 Clip against $\overline{BC}$.

5.17 Use the Weiler–Atherton algorithm to clip the polygon in Fig. 5-29(*a*).

SOLUTION

See Fig. 5-29(*b*) and (*c*).

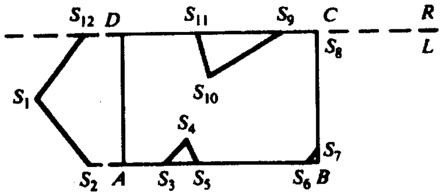


Fig. 5-26 Clip against $\overline{CD}$.

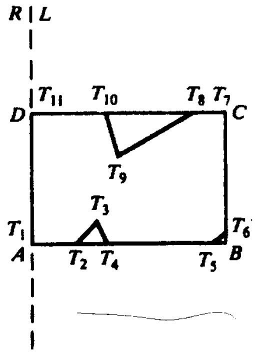
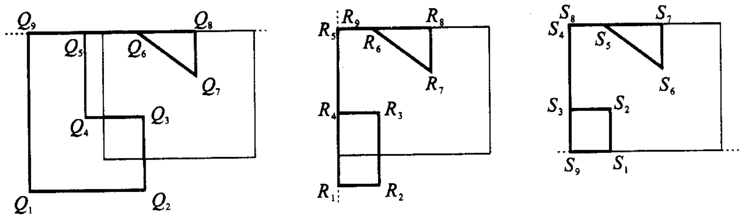


Fig. 5-27 Clip against $\overline{DA}$.

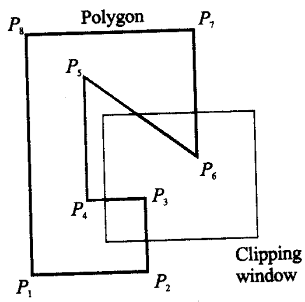


Clip against top boundary

Clip against left boundary

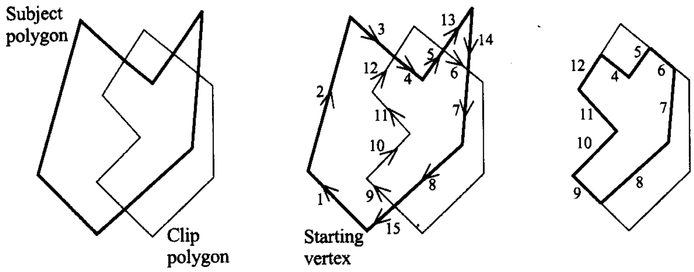
Clip against bottom boundary

Fig. 5-28



(*a*)

Fig. 5-10



(*a*) (*b*) (*c*)

Fig. 5-29